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Mean-Variance Versus Direct Utility Maximization

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ABSTRACT

Levy and Markowitz showed, for various utility functions and empirical returns distributions, that the expected utility maximizer could typically do very well if he acted knowing only the mean and variance of each distribution. Levy and Markowitz considered only situations in which the expected utility maximizer chose among a finite number of alternate probability distributions. The present paper examines the same questions for a case with an infinite number of alternate distributions, namely those available from the standard portfolio constraint set.

IT IS FREQUENTLY ASSERTED that mean-variance analysis applies exactly only when distributions are normal or utility functions quadratic, suggesting that it gives almost optimum results only when distributions are approximately normal or utility functions look almost like a parabola. On the other hand, in a recent paper Levy and Markowitz [6] showed empirically that the ordering of portfolios by the mean-variance rule was almost identical to the order obtained by using expected utility for various utility functions and historical distributions of returns.

For example, the authors calculated an "exact" expected utility (of, say, a logarithmic utility function) for each of 149 mutual funds by attributing an equal probability for each year in the sample. Using the same data, the expected utility was approximated by a function of mean and variance $U = f(E, V)$, where E represents the mean and V represents the variance. The exact expected utility and the approximation based only on E and V were found to be highly correlated. The analysis was repeated for various sets of data and various utility functions, and the same results were obtained in almost every case.

Good results were obtained, then, when EU and $f(E, V)$ were compared for a finite number of portfolios, e.g., 149 mutual funds. However, it may be that $f(E, V)$ would do less well when asked to find the best portfolio among the infinite number of possible mixtures of a finite number of securities. In this case, the exact maximizing of expected utility might lead to quite different results than those obtained by using the mean-variance approximation. The aim of the present paper is to compare the expected utility of the optimum portfolio for given utility functions with the expected utility of well-selected portfolios from the mean-

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variance efficient frontier. Specifically, for various probability distributions and utility functions:

(a) We derive a fine mesh of points along the mean-variance efficient frontier, and calculate the expected utility of all these portfolios. From these we select the portfolio with the highest expected utility.
(b) Using the same set of data of individual stocks, we select the portfolio which maximizes expected utility of the given utility function—not just maximum among mean-variance efficient portfolios, but among all feasible portfolios.

Comparison of the expected utilities obtained in (a) and (b) indicates the possible error in investment decision made using the mean-variance framework.

We also compare the portfolios obtained in (a) and (b). In addition, we compare the expected utility of the optimum portfolio with the expected utilities of portfolios containing equal weights of a few securities with highest means. Finally, we examine the effect of leverage on the approximation of the E-V maximization to the direct utility maximization.

The present paper is similar to that of Pulley [9] who also compares mean-variance efficient portfolios which approximately maximize expected utility with portfolios which actually maximize expected utility. There are two principal differences between the results presented here and those of Pulley. The most important difference between Pulley's and the present work is the criteria used to compare the expected utility EU_M achieved by the mean-variance approximation with the actual maximum expected utility EU_A. Pulley made an error in choice of criterion which completely invalidates his comparisons. Recall that if $U(R)$ is a utility function then, for any real number c, $c + U(R)$ is another utility function with exactly the same ranking of probability distributions. Pulley's ratio criteria EU_M/EU_A is not invariant to the choice of c. In fact, it can always be made arbitrarily close to 1.0 (the best possible score) or near to zero, or even negative by choice of the irrelevant additive constant.

For example, suppose someone tested a great many utility functions including, among others, $U(R)$, $U(R) + 1000$, and $U(R) - 5$. Of course, what we call $U(R)$ and what we call $U(R) + 1000$ is arbitrary—an accident of history. They are equivalent utility functions. Suppose that the portfolio which maximizes expected utility happens to have $EU(R) = 8$ while the mean-variance approximation happens to have $EU(R) = 4$. This gives us a mediocre Pulley score of $4/8 = 0.5$. But the equivalent utility function, $U(R) + 1000$, has the great Pulley score of $1004/1008 = 0.99+$; while the equally equivalent utility function $U(R) - 5$ has the terrible Pulley score of $-1/3 = -0.33$.

In general, equivalent utility functions can always be made to give arbitrarily good or arbitrarily bad Pulley scores. Pulley's results should, therefore, be viewed like the results of an experiment on heat made with a broken thermometer—not just a slightly inaccurate thermometer, but a capricious one that can read freezing or boiling for two bodies with the same temperature.

Another difference between the present article and the Pulley article is that the latter is concerned, as its title states, with "short holding periods." Accordingly, Pulley's analyses deal with monthly and semiannual holding periods. The analyses in the present paper, on the other hand, use annual holding periods. As both Pulley and Levy-Markowitz explain, the higher the portfolio variance the less likely is a mean-variance approximation to do almost as well as actual expected utility maximization. Thus, our use of annual data poses a greater challenge for mean-variance than do Pulley's monthly and semiannual analyses. We further challenge mean-variance by including an analysis which allows borrowing.

I. The Problem

A portfolio is mean-variance efficient if it maximizes expected rate of return (E) for a given variance (V), and minimizes the variance for a given expected return. Let us denote by X_i the proportion of the ith asset in the portfolio. Thus, assuming the standard constraint set without borrowing, an efficient portfolio $\underline{X}' = (X_1, X_2, \cdots, X_n)$ solves the following problem:

$$\text{Minimize } \underline{X}' \textstyle\sum \underline{X}$$

subject to

$$X_i \geq 0 \quad i = 1, 2, \cdots, n$$

$$\underline{X}'\underline{M} = E$$

and

$$\underline{X}'1 = 1$$

where $\sum$ is the covariance matrix, M is the vector of mean returns of the n securities, E is the mean return of the portfolio, and 1 represents either the number 1 or a vector of 1's as needed.

For various values of E we obtain various efficient portfolios. For each E-V efficient portfolio, one may calculate its expected utility.

$$EU(\textstyle\sum_{i=1}^{n} X_i R_i)$$

where R_i is the return on the ith security. In principle, by calculating expected utility for all efficient portfolios, one can select the E-V efficient portfolio which maximizes expected utility. The maximum expected utility obtained on the E-V efficient set will be denoted by $E^*U(\ )$.

While $E^*U(\ )$ is the solution to the maximization of the expected utility over the set of the E-V efficient set, the optimal portfolio is obtained by allowing all possible investment mixes, and not only the E-V efficient portfolios. The optimal portfolio is given by solving the following optimization problem:

$$\underset{X}{\text{Max}}\ EU(\textstyle\sum_i X_i R_i)$$

subject to

$$X_i \geq 0 \quad i = 1, 2, \cdots, n$$

and

$$\underline{X}'1 = 1$$

The value obtained by this maximization, which we shall refer to as the direct maximization, will be denoted by $EU(\ )$ as distinguished from $E^*U(\ )$.

Since $EU(\ )$ is a general maximization without the constraint that $\underline{X}$ be E-V efficient, $E^*U(\ ) \leq EU(\ )$. Finding $\underline{X}$ which solves for $EU(\ )$ (direct maximization) is not a trivial exercise even by computer. It has the following disadvantages:

(a) It requires a large number of calculations, typically several times the number necessary to trace out the E-V efficient frontier. More details on the method of direct maximization are given in the Appendix.

(b) In calculating $EU(\ )$ for a given set of data, one has to repeat all the calculations for each of the various U's which one considers. In contrast, the E-V efficient portfolios can be found once for all utility functions leaving the choice of $E^*U(\ )$ from the E-V frontier a relatively minor task.

Consider an investment consultant who would like to find his customers' optimal investment strategies. However, he cannot do so without knowledge of the investors' particular utility functions. Moreover, he has many clients who may differ with respect to their preferences. On the other hand, if $E^*U(\ )$ is almost equal to $EU(\ )$, the consultant can overcome the difficulty of not knowing the investors' preferences simply by deriving the E-V efficient portfolios and presenting only these alternative porfolios to his customers. Each investor would choose from the E-V efficient set a portfolio according to his particular preferences, which may not be explicitly stated.

We must analyze, however, the loss of welfare incurred by using $E^*U(\ )$ as the optimal criterion rather than $EU(\ )$. The remainder of this paper is primarily concerned with the development and measurement of an index of this welfare loss. We shall test the approximation of $E^*U(\ )$ to $EU(\ )$ for certain frequently cited utility functions.

II. The Quality of the Approximation

One could measure the loss of utility incurred by choosing among E-V efficient portfolios by the difference $D = EU(\ ) - E^*U(\ )$; but, D is not invariant to linear transformations of the utility functions. A natural choice instead for an index is:

$$I = \frac{E^*U(\ ) - E_N U(\ )}{EU(\ ) - E_N U(\ )}$$

where $E_N U(\ )$ is the expected utility of a "Naive" portfolio in which $\frac{1}{n}$ is invested in each security, namely:

$$E_N U(\ ) = EU\left(\sum_{i=1}^{N} \frac{1}{n} R_i\right)$$

By definition, I is less than one. It can be negative, but one would not expect the best E-V portfolio to be worse than a naive portfolio; thus, in most cases $0 \leq I \leq 1$. If I is close to zero, we can conclude that the E-V efficiency criterion is not very promising, since the naive method gives almost the same expected utility.

When the index I is close to one, the approximation is good and the error in using the E-V criterion is small.

Another possible index for measuring the welfare loss is:

$$I_R = \frac{E^*U(\ ) - E_R U(\ )}{EU(\ ) - E_R U(\ )}$$

where $E_R U(\ )$ is the expected utility of a portfolio selected at random from a uniform distribution, i.e., with every subset of the constraint set having the same probability of including the selected portfolio as any other subset of equal volume. We repeat the random selection several times and calculate the average expected utility across all random portfolios. In fact, we found this average almost equal to $E_N U(\ )$. Thus, we report only the results for the index I.

III. The Selected Utility Functions

The following utility functions are used in the empirical tests:

$-e^{-(1+R)}$
$(1 + R)^a$, $(a = 0.1, 0.5, 0.9)$
$(2 + R)^a$, $(a = 0.1, 0.5)$
$\ln(i + R)$, $(i = 1, 2)$

where R is defined as the rate of return on investment. All of these functions have $U' > 0$, $U'' < 0$, and $U''' > 0$. In Table I we list the properties of these functions with respect to the absolute and relative risk aversion measures of Arrow [1] and Pratt [8]. Note that in Table I, W corresponds to $R + 1$ above. For example, $(2 + R)^a = (B + W)^a$ where $B = 1$.

IV. The Data

We selected three mutually exclusive samples of 10, 12, and 20 stocks from the CRSP tape. Since the results are very similar, we report here only the results of the 20-stock sample. It is not that we recommend past history alone as a predictor

Table I

Some Properties of the Selected Utility Functions Defined on Wealth, W

Utility	Absolute Risk-Aversion Measure	Proportional Risk-Aversion Measure
$-l^{-\alpha W}$, $(\alpha > 0)$	Constant	Increasing
$(W + B)^\alpha$, $(0 < \alpha < 1)$	Decreasing	Increasing for $B > 0$
		Constant for $B = 0$
		Decreasing for $B < 0$
$\ln(W + B)$, $(B > 0)$	Decreasing	Increasing if $B > 0$
		Constant if $B = 0$
		Decreasing if $B < 0$

of future returns. Rather we use this data as examples of real world security and portfolio moments.

In Table II, we present the means, standard deviations, coefficient of variation, the relative skewness, and the kurtosis of the annual returns of these 20 stocks in the years 1949–1968.

The question has been raised concerning the Levy-Markowitz results whether the ability of $f(E, V)$ to approximate $EU(R)$ was due to normality of the underlying distributions rather than the asserted robustness of the quadratic approximation. For the present data, Table II clearly rejects the notion that the return distributions are normal. First note that only one security is negatively skewed. If these were independent drawings from any symmetric distribution, then the probability would be only $21/2^{20} \approx 0.0002$ that only zero or one sample would be negatively skewed. The significance of this observation is clouded by the lack of independence between securities. Recall that if returns were normal then the coefficient of skewness would be roughly normal with mean 0 and standard deviation $= \sqrt{6/N} = 0.55$, and the deviation of the coefficient of kurtosis from 3 would be roughly normal with mean 0 and standard deviation $= \sqrt{24/N} = 1.1$ (see Kendall and Stuart [5]). In Table II, 6 out of the 20 securities have

Table II

The Average Rate of Return, Standard Deviation, Coefficient of Variance, and Relative Skewness of the 20-Stock Sample in the Years 1949–1968

Stock	Means %	Standard Deviations %	Coefficient of Variation*	Relative Skewness**	Kurtosis***
1. Conelco	30.67	100.24	3.27	2.97	10.17
2. Texas Gulf	23.50	59.36	2.53	2.61	8.93
3. Carpenter	23.25	37.19	1.60	0.71	0.705
4. Cerro	21.38	44.51	2.08	0.55	−1.00
5. Chrysler	20.49	50.31	2.45	2.31	7.32
6. California Pack.	20.30	24.22	1.19	0.26	0.78
7. Dana Co.	20.04	30.26	1.50	1.03	1.52
8. Sterling Drugs	17.55	17.34	0.99	−0.49	1.11
9. Copperweld Steel	17.53	38.27	2.18	1.17	2.57
10. Crucible Steel	16.95	38.48	2.27	0.42	−0.08
11. Mobil Oil	16.87	23.69	1.40	0.04	−0.94
12. Colt	15.15	43.76	2.89	0.50	0.26
13. Standard Oil	14.82	19.26	1.30	0.22	−0.68
14. Sucrest Co.	14.20	29.33	2.06	0.56	1.66
15. Sunray	14.14	24.18	1.68	0.50	−0.68
16. Chemway	13.84	35.65	2.58	0.84	0.18
17. Continental Can	13.53	15.23	1.13	0.21	−0.58
18. Detroit Steel	13.45	32.44	2.41	0.37	−0.56
19. Spartan	12.84	44.82	3.49	1.21	2.15
20. City Stores	9.10	24.62	2.71	1.96	4.49

* Coefficient of Variation is the standard deviation over the mean.

** Relative skewness is measured by $\mu_3/(\mu_2)^{3/2}$ where μ_2 and μ_3 are the second and third central moments, respectively.

*** Kurtosis is defined here as $(\mu_4/\mu_2^2) - 3$ where μ_4 is the fourth central movement.

observed skewness at least 2 standard deviations from zero, and some of these are 4 or 5 standard deviations from expected under the normal hypothesis. Furthermore, several have observed kurtosis which is 4 to 9 standard deviations from expected. Clearly, not all security returns are normal.

V. The Empirical Results

Table III presents the optimum portfolios selected by direct maximization for various utility functions. From the table we see that:

> Ignoring the small proportion invested in Chrysler in the case of the negative exponential utility function, only 4 securities out of the available 20 securities appear with positive proportions in the optimal portfolio.

The 4 selected securities are from the group of 6 securities with the highest mean (see Table II).

Conelco, with highest mean, always appears in the optimal portfolio. The securities, Texas Gulf and Carpenter, with second and third highest mean, almost always appear in the optimal portfolio. The fourth highest (Cerro) never appears in the optimal portfolio; and the fifth (Chrysler) almost never appears. California Pack. which is ranked only sixth according to mean is selected in many cases, perhaps due to its low variance. (California Pack. has almost the lowest coefficient of variation.)

In order to compare the above with E-V efficient portfolios, we first derived a "mesh" of E-V efficient portfolios with mean ranging from 16.5 to 30.3 by steps of 0.2. Table IV presents a selection of portfolios from this set. The table reports for each portfolio, the mean, the standard deviation, and the investment allocation. Note that, like the direct maximization, the E-V efficient portfolios are not well-diversified. Out of the 20 securities, only 9 are ever used. Moreover, for much of the efficient frontier only 4–5 stocks account for 100% of the portfolios.

We next calculated the expected utility of each portfolio in our mesh of E-V efficient portfolios. Since the investment allocation is given, no maximization is

Table III

Optimal Investment Strategies with a Direct Utility Maximization

Utility Function	California (6)*	Carpenter (3)	Chrysler (5)	Conelco (1)	Texas Gulf (2)	Total	Average Return	Standard Deviation
$-e^{-x}$	44.3	34.7	0.2	5.5	15.3	100%	22.4	27.3
$X^{0.1}$	33.2	36.0	-	13.6	17.2	100%	23.3	32.3
$X^{0.5}$	—	42.2	—	34.4	23.4	100%	25.9	49.4
$X^{0.9}$	—	—	—	97.6	2.4	100%	30.5	98.6
$\ln(X)$	37.9	34.8	—	11.1	16.2	100%	23.1	29.4
$\ln(X + 1)$	3.7	46.8	—	26.1	23.4	100%	25.1	43.7
$(X + 1)^{0.1}$	0.4	44.9	—	30.3	24.4	100%	25.5	46.7
$(X + 1)^{0.5}$	—	33.5	—	66.5	—	100%	28.2	74.2

* Numbers indicate rank according to mean, Table II.

Table IV

Proportions of Stocks Average Return, Standard Deviation of Portfolios on the *E*-*V* Efficient Frontier, and the Implied Risk-Free Rate.*

Security						
California	9.46	13.62	16.40	21.45	25.13	30.65
Carpenter				3.46	6.27	10.47
Chrysler				0.46	1.00	1.81
Continental	32.84	25.41	20.46	15.28	17.35	7.97
Dana	5.85	7.96	9.36	6.91	4.55	1.02
Mobil Oil	0.62	0.47	0.37	1.67	2.41	3.59
Sterling Drugs	37.32	39.25	40.53	39.87	38.84	37.29
Sucrest Co.	10.33	8.50	7.28	4.82	3.12	0.58
Texas Gulf	3.58	4.79	5.60	6.09	6.30	6.62
TOTAL	100.00	100.00	100.00	100.00	100.00	100.00
Average Return	16.5	17.10	17.50	18.10	18.50	19.10
Standard Deviation	11.70	12.51	13.19	14.36	15.70	16.53
Risk-free Rate	6.66	9.47	10.50	11.19	11.51	11.90

Security						
California	34.31	38.99	41.91	46.08	48.55	46.04
Carpenter	12.73	15.45	19.70	25.86	29.53	35.07
Chrysler	2.39	3.08	2.80	2.38	2.13	0.90
Conelco				0.24	0.67	2.62
Continental	5.18	0.21				
Mobil Oil	2.41					
Sterling Drugs	35.69	33.65	25.66	13.56	6.00	
Texas Gulf	7.28	8.62	9.94	11.88	13.12	15.36
TOTAL	100.00	100.00	100.00	100.00	100.00	100.00
Average Return	19.5	20.10	20.50	21.10	21.50	22.10
Standard Deviation	17.46	18.91	19.97	21.77	23.08	25.23
Risk-free Rate	12.17	12.70	13.53	14.39	14.74	16.04

Security					
California	39.37	28.50	21.29	10.39	3.15
Carpenter	38.59	43.38	46.57	51.36	54.56
Conelco	4.97	8.65	11.11	14.80	17.26
Texas Gulf	17.08	19.47	21.06	23.45	25.04
TOTAL	100.00	100.00	100.00	100.00	100.00
Expected Return	22.50	23.10	23.50	24.10	24.50
Standard Deviation	26.96	29.97	32.20	35.80	38.34
Risk-free Rate	16.85	17.64	18.00	18.39	18.59

Security						
Carpenter	50.07	44.55	36.28	30.76	22.49	16.97
Conelco	24.06	29.45	37.53	42.91	50.99	56.38
Texas Gulf	25.87	26.00	26.19	26.32	26.52	24.65
TOTAL	100.00	100.00	100.00	100.00	100.00	100.00
Expected Return	25.10	25.50	26.10	26.50	27.10	27.50
Standard Deviation	42.45	45.59	50.78	54.48	60.31	64.34
Risk-free Rate	19.57	20.05	20.55	20.79	21.08	21.23

Security						
Carpenter	8.70	3.18				
Conelco	64.46	69.85	78.10	83.68	92.05	97.63
Texas Gulf	26.84	26.97	21.90	16.32	7.95	2.37
TOTAL	100.00	100.00	100.00	100.00	100.00	100.00
Expected Return	28.10	28.50	29.10	29.50	30.10	30.30
Standard Deviation	70.56	74.79	81.33	85.92	93.12	98.09
Risk-free Rate	21.41	21.51	21.94	22.22	22.56	22.66

* This implied risk-free rate is the rate at which a straight line which is tangent to the efficient risky frontier crosses the vertical axis. The risk-free rate obtained here is a technical result which is not necessarily equal to the actual risk-free rate.

needed. For example, for the first E-V efficient portfolio in Table IV we calculated a rate of return for each period t,

$$R_t = 0.0946R_{1t} + 0.3284R_{2t} + 0.058R_{3t} + 0.00062R_{4t} + 0.3732R_{5t} + 0.1033R_{6t} + 0.0358R_{7t}$$

where R_{it} is the annual rate of return in year t of the ith security included in the efficient portfolio. From this we calculate the expected utility

$$EU(\ ) = \frac{1}{20} \sum_{t=1}^{20} U(R_t).$$

For each given utility function, we repeat this calculation for all efficient portfolios and then select the E-V efficient portfolio which yields the highest expected utility, $E^*U(\ )$.

For example, the portfolio in our mesh of E-V efficient portfolios with the highest expected utility for $U = \ln(1 + R)$ is that which yields 24.7 percent with a standard deviation of 39.6 percent (see Table V below). There may exist a better E-V efficient portfolio in the neighborhood of this mesh point; thus, our results underestimate the E-V approximation.

Table V presents the mesh portfolios with highest expected value for various utility functions. A comparison of the investment strategies given in Tables III and V shows that the direct maximization of $EU(\ )$ and maximization of $E^*U(\ )$ often results in quite similar portfolios. Even if the investment allocations differ, the performance index I may be close to 1.0, indicating a small welfare loss from maximizing $E^*U(\ )$ rather than $EU(\ )$.

Table VI shows the derived expected utility as well as the values of the index I. The results are quite impressive. The index is equal or close to one in all cases. The lowest index of 0.978 is obtained in the case of $\ln(X)$. This implies that these risk-averse investors would lose almost nothing by selecting the optimum portfolio from an E-V efficient set rather than using direct maximization over all feasible portfolios considered.

Figure 1 presents the E-V efficient frontier, the location of the naive portfolio (denoted by "N"), the portfolios with a direct maximization of $EU(\ )$ (denoted by "O"), and the portfolios which maximize EU among E-V efficient portfolios (denoted by "×"). By definition the direct utility function cannot lie to the left

Table V

Optimum E-V Portfolios for Various Utility Functions

Utility Function	California (6)	Carpenter (3)	Conelco (1)	Texas Gulf (2)	Total	Average Return	Standard Deviation
$-e^{-X}$	39.4	38.6	5.0	17.0	100%	22.5	27.0
$X^{0.1}$	28.5	43.4	8.6	8.6	100%	23.1	30.0
$X^{0.5}$	—	41.8	32.1	26.1	100%	25.7	47.3
$X^{0.9}$	—	—	97.6	2.4	100%	30.5	98.1
$\ln(X)$	32.9	41.8	7.4	18.7	100%	22.9	28.9
$\ln(X+1)$	—	55.6	18.7	25.7	100%	24.7	39.6
$(X+1)^{0.1}$	—	50.1	24.0	25.9	100%	25.7	47.3
$(X+1)^{0.5}$	—	3.2	69.8	27.0	100%	28.5	75.8

Table VI

Direct and Approximated Expected Utility and the Approximation Index

Utility Function	The Expected Utility from Direct Maximization $EU(\ )$	The Highest Utility from an E-V Efficient Portfolio $E^*U(\ )$	Expected Utility of the Naive Portfolio $EU_N(\ )$	Approximation Index* I
$-e^{-X}$	−0.30382	−0.30390	−0.31610	0.993
$X^{0.1}$	1.01842	1.01838	1.01466	0.989
$X^{0.5}$	1.10465	1.10459	1.07940	0.998
$X^{0.9}$	1.24933	1.24934	1.15482	1.000
$\ln(X)$	0.18016	0.17935	0.14387	0.978
$\ln(X+1)$	0.79575	0.79572	0.77215	0.999
$(X+1)^{0.1}$	1.08300	1.08300	1.08033	1.000
$(X+1)^{0.5}$	1.49668	1.49664	1.47308	0.998

* The Approximation Index is given by:

$$I = \frac{E^*U(\ ) - E_N U(\ )}{EU(\ ) - E_N U(\ )}$$

where $E_N U(\ )$ assumes investment of $\frac{1}{n}$ in each security.

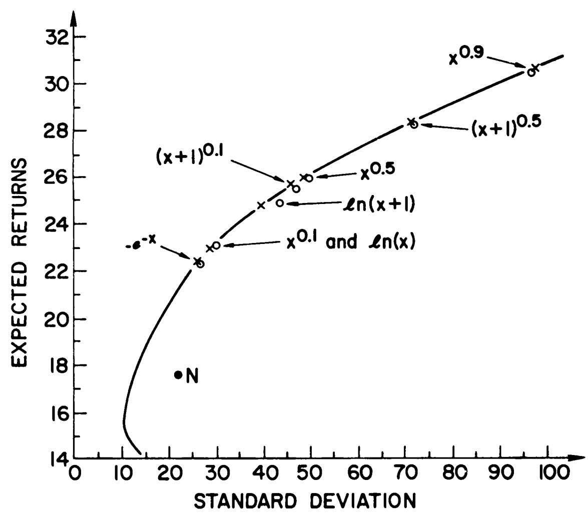

Figure 1. E and σ of Various Portfolios

of the efficient frontier; but may lie on or to the right of the frontier. In fact, all points "O" lie almost on the efficient set. This comparison is given numerically in Table VII. Again we see that the E and σ of both maximization methods are very much the same and that the direct maximization portfolios have E's and σ's which are almost on the efficient E-V frontier. The optimal portfolios from direct maximization have a standard deviation which is higher only by 3.8–0.0% than the minimum standard deviation which can be obtained on the *E-V* frontier for the same mean. These deviations from the *E-V* frontier are relatively small in comparison to the naive portfolio which has a standard deviation which is 70.5% greater than that of the *E-V* portfolio with the same mean of 17.5%.

Previously we noted that the optimal portfolios usually contain 3 or 4 securities, where these securities tend to have the highest mean return. It is tempting to conjecture, therefore, that holding equal amounts of 2 to 5 securities with the highest mean may yield almost optimum results. Table VIII tests this hypothesis. The performance index (I) of the *E-V* criterion is compared with the indexes of

Table VII

E and σ of Optimal Portfolios According to *E-V* and Direct Maximization Methods, Proximity of Direct Maximization Optimal Portfolios to *E-V* Efficient Frontier

	E and σ of Optimal Portfolios					
Utility Function (1)	*E-V* Maximization *E* (2)	σ (3)	Direct Maximization *E* (4)	σ (5)	$(E\text{-}V)^*$ (6)	$\frac{\sigma-\sigma(E\text{-}V)}{\sigma(E\text{-}V)}$ (7)
$-e^{-X}$	22.5	27.0	22.4	27.3	26.5	3.0%
$X^{0.1}$	23.1	30.0	23.3	32.3	31.1	3.8%
$X^{0.5}$	25.7	47.3	25.9	49.4	49.0	0.8%
$X^{0.9}$	30.5	98.1	30.5	98.6	98.1	0.5%
$\ln(X)$	22.9	28.5	23.5	29.9	29.9	0.0%
$\ln(X+1)$	24.7	35.6	25.1	43.7	42.5	2.8%
$(X+1)^{0.1}$	25.7	47.3	25.5	46.7	45.6	2.4%
$(X+1)^{0.5}$	28.5	75.8	28.2	74.2	71.6	3.6%
The naive portfolio of equal proportions			17.5	22.5	13.2	70.5%

* $\sigma(E\text{-}V)$ denotes the standard deviation of portfolios on the efficient *E-V* frontier that have the same mean as the mean of the optimal portfolios obtained by the direct maximization.

Table VIII

The Indexes (I) of *E-V* Portfolios and Portfolios of Equal Proportions of 1–5 Stocks with the Highest Means

		Portfolio with Equal Proportions of K Stocks with Highest Mean				
Utility Function	*E-V* Portfolio	$K = 1$	$K = 2$	$K = 3$	$K = 4$	$K = 5$
$-e^{-X}$	0.993	−2.375	−0.321	0.400	0.405	0.512
$X^{0.1}$	0.989	−0.823	0.400	0.791	0.703	0.726
$X^{0.5}$	0.998	0.506	0.888	0.989	0.872	0.820
$X^{0.9}$	1.000	1.000	0.798	0.716	0.623	0.555
$\ln(X)$	0.978	−1.178	0.234	0.693	0.615	0.659
$\ln(X+1)$	0.999	0.209	0.812	0.979	0.875	0.833
$(X+1)^{0.1}$	1.000	0.379	0.863	0.989	0.882	0.832
$(X+1)^{0.5}$	0.998	0.913	0.955	0.940	0.827	0.755

portfolios with equal proportions of securities with the k highest means, for $k = 1, \cdots, 5$.

In most cases the approximation index of the equal proportions portfolio of the highest mean stocks is much lower than the index of the best E-V efficient portfolio. In some cases the former index is even negative. In particular, holding only one stock with the highest mean will yield very poor and even negative approximation indexes in almost all cases with the exception of the $X^{0.9}$ function where the investor is almost risk neutral.

Thus, using only the mean, ignoring the variances and covariances, in order to construct equal proportions portfolios of 2 to 5 stocks frequently improves expected utility relative to the naive portfolio of equal proportions of all stocks. However, using both the portfolio mean and variance to select portfolios will usually improve expected utility much more. Additional moments besides the mean and variance will not improve the approximation substantially, as the approximation index of the E-V criterion is really very close to one.

VI. The Effect of Leverage

Leverage increases the risk of the portfolio. If the investor borrows part of the funds invested in the risky portfolio, then the fluctuations of the return on these leveraged portfolios will be proportionately greater. If unlimited leverage is allowed the quadratic approximations described in [6] and [7] can be forced to fail, e.g., since losses approaching 100% will give an expected logarithm approaching $-\infty$ while the quadratic remains finite. In this section, we examine the effect of limited leverage on the approximation index. Specifically we restricted the amount of borrowing to a maximum of 50% of the investment.

Assuming a 10% risk-free rate,[1] we derived a mesh of portfolios on the efficient E-V frontier for the n securities plus an n + 1st risk-free security allowed to take negative values constrained by 50%. For each utility function, we calculated the expected utility of each of these efficient portfolios, and selected the portfolios which maximized expected utility. In addition, we obtained the true optimum portfolios using exactly the same basic technique as in the unleveraged case but allowing 50% borrowing.

Thus the direct maximization process solved the following problem:

$$\text{Max } EU\left(\textstyle\sum_{i=1}^{n} X_i R_i + \left(1 - \sum_{i=1}^{n} X_i\right) R_F\right)$$

subject to $X_i \geq 0 \quad i = 1, 2, \cdots, n$

and $X_{n+1} = 1 - \sum_{i=1}^{n} X_i \geq -0.5$

where X_{n+1} is the proportion invested in the risk-free asset.

The index of approximation requires the choice of a "naive" portfolio. In the previous case, without leverage, we simply let $X_i = 1/n$ for each i. In the present case we use a leveraged naive portfolio, i.e., the solution to the following:

$$\text{Max } U\left[(1 - X_{n+1}) \textstyle\sum_{i=1}^{n} R_i/n + X_{n+1} R_F\right]$$

[1] When we repeated the analysis with a 7% rate, we got similar results.

subject to $-0.5 \leq X_{n+1} \leq 1$, where X_{n+1} is the proportion invested in the riskless asset with yield R_F. Thus the "naive" portfolio was allowed optimum leverage.

Table IX presents the approximation index for the E-V maximization. Once again the approximation indexes are high; indeed, about the same as the indexes in the unleveraged case. At least in our empirical sample, then, leverage limited to 50% does not lead to perceptibly worse approximation indexes. Notice that, with the risk-free rate of 10%, the maximum amount of borrowing of 50% was used in all cases. Even the naive portfolio, with a mean return of 17.5%, is leveraged by 50% for all utility functions.

VII. Conclusions

Levy and Markowitz reported that, for various finite populations of portfolio returns such as the returns on 149 investment company portfolios, the best mean-variance efficient portfolio was frequently the portfolio which maximized expected utility—or at least had a near optimum expected utility. The present paper reports that, for various utility functions and the historical returns on 3 different sets of securities, when a portfolio may be chosen from any of the infinite number of portfolios of the standard constraint set the best mean-variance efficient portfolio has almost maximum obtainable expected utility. This remained true when 50% borrowing was allowed.

The hypothesis was tested, and rejected, that similar excellence could be obtained by investing equally in the k securities with highest expected returns, for k equal to about the number of securities in the optimum portfolio. It was also found that the excellence of the mean-variance result was not due to normality of data. Rather it illustrates the robustness of the quadratic approximation as reviewed by Levy and Markowitz.

Table IX

Direct and Approximated Expected Utility with Borrowing and Lending at Risk-Free Interest* and the Approximation Index

Utility Function	Optimum Leverage %	Expected Utility of Direct Maximization $EU(\ )$	Approximated Expected Utility with E-V $E^*U(\ )$	Expected Utility of Naive Portfolio $E_NU(\ )$	Approximation Index I**
$-e^{-X}$	50	−0.29633	−0.29641	−0.31333	0.996
$X^{0.1}$	50	1.02105	1.02100	1.01593	0.990
$X^{0.5}$	50	1.12341	1.12258	1.09110	0.973
$X^{0.9}$	50	1.31880	1.31847	1.18642	0.998
$\ln(X)$	50	0.20384	0.20303	0.15370	0.983
$\ln(X+1)$	50	0.81392	0.81352	0.78345	0.987
$(X+1)^{0.1}$	50	1.08504	1.08499	1.08162	0.986
$(X+1)^{0.5}$	50	1.51393	1.51389	1.48374	0.999

* The risk-free rate is 10% and the amount of leverage is constrained by 50%.

** $I = \dfrac{E^*U(\ ) - E_NU(\ )}{EU(\ ) - E_NU(\ )}$.

Appendix

The Direct Maximization Algorithm[2]

A. The Problem

There are n risky options and m periods. Let X_{ij} be the return on security i in period j. Let u be a utility function with $u' \geq 0$ and $u'' \leq 0$. The maximization problem is:

$$\text{Max}\left\{\frac{1}{m}\sum_{j=1}^{m} u\left(\sum_{i=1}^{n} y_i X_{ij}\right)\right\}$$

subject to $\sum_{i=1}^{n} y_i = 1$.

Let us first solve the optimization problem for the case where short sales are not allowed. Thus, we add the constraints $y_i \geq 0$ for $i = 1, \cdots, n$. In order to switch from the inequality constraints $y_i \geq 0$ to equality constraints, we use the following method. Redefine y by $z_i^2 = y$, and let us denote

$$f(z) \equiv -\sum_{j=1}^{m} u\left(\sum z_i^2 X_{ij}\right)$$

and the constraint will be:

$$g(z) \equiv \sum_{i=1}^{n} z_i^2 - 1 = 0.$$

Let us use an augmented-Lagrangian method,[3] where

$$L(z, \lambda, r) = f(z) + \lambda g(z) + \tfrac{1}{2} r \times g(z)^2$$

and the maximization constrained problem will be:

$$\text{Min } L(z, \lambda, r) \qquad \text{subject to} \quad g(z) = 0.$$

We use Powell's Hestenes Algorithm for finding the solution of this problem[4] which is described below.

Step 1 Determine r_0, λ_0, z_0 for $K = 0$. In our cases we determined $r_0 = 10$, $\lambda_0 = 1$, $z_0^i = \frac{1}{n}$, $i = 1, \cdots, n$.

Step 2 Find z_K such that: $L(z_K, \lambda_K, r_K) = \underset{z}{\text{Min}}(z, \lambda_K, r_K)$. We selected the Fletcher and Powell Method[5] to find this unconstrained minimum. The method will be presented later with more details.

Step 3 Updating by: $\lambda_{K+1} = \lambda_K + r_K g(z_K)$. (This is the updating formula that was proposed by Hestenes and Powell—see Hestenes [4].)

Step 4 Updating $r_{K+1} = \tau r_K$ where $\tau = 1.1$. The size of $\tau = 1.1$ was determined by us empirically after a few trials.

Step 5 Checking the stopping criterion.

$$\| \nabla_Z L(z_K, \lambda_K, r_K) \| + \| g(z_K) \| < \epsilon$$

[2] This section was written, and the application of this algorithm to our problem was carried out, by Ammon Gonan from Technion, Institute of Technology in Haifa, Israel.

[3] See Hestenes [4].

[4] See Hestenes [4].

[5] See Fletcher and Powell [3].

where $\nabla_Z L(,,)$ is the gradient of L with respect to z. We selected $\epsilon = 10^{-3}$.[6] If the criterion is not satisfied, define $K = K + 1$ and go back to *Step 2.*

B. Details on the Maximization Method of Step 2

The Fletcher and Powell Method which was selected belongs to the Quasi-Newton algorithms group. The method is in the IMSL library and a description of the method is given in textbooks of nonlinear search methods (see for example, Avriel [2] pp. 322–34).

The F & P Algorithm for Min $f(x)$ is given by the following steps.

$$\text{Denote:} \quad P^K = x^K - x^{K-1}$$
$$\gamma^K = \nabla f(x^K) - \nabla f(x^{K-1})$$

Step 1 Given x_0, $\nabla f(x_0)$, and an arbitrary symmetric $n \times n$ positive definite matrix H_0. (For example $H_0 = 1$.) Initially $K = 0$.

Step 2 Find $\bar{\lambda}$ such that:

$$f(x_K - \bar{\lambda} H_K \nabla f(x_K)) = \min_{\lambda} f(x_K - \lambda H_K \nabla f(x_K))$$

and define $Y_{K+1} = x_K + \bar{\lambda} H_K \nabla f(x_k)$.

Step 3 Calculate ∇, P^{K+1}, γ^{K+1}, and define

$$H_{K+1} = H_k + \frac{P^{K+1}(P^{K+1})^T}{(P^{K+1})^T,\ \gamma^K} - \frac{(H_K \gamma^{K+1})(H_K g^{K+1})T}{(\gamma^{K+1})^T H_K \gamma^{K+1}}$$

H_K is an approximation to the inverse of the Hessian of t, i.e., $H_K \approx (\nabla^2_{xx} f(x_K))^{-1}$.

Step 4 Check a stopping rule otherwise go back to Step 2. The stopping rule is again $\| \nabla f(x_K) \| < \epsilon$.

[6] Stopping rules of $\epsilon = 10^{-6}$ and $\epsilon = 10^{-8}$ were also examined. The results were not meaningfully different from the results obtained by $\epsilon = 10^{-3}$.

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