



The Sortino Ratio and Darsinos and Satchell's 'Generalised Sharpe Ratios' Fail the Lottery Test

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1 Introduction

Ranking investment decisions requires ranking distributions. This requires a functional on distributions, i.e. a real-valued function which produces a number for each distribution. The Sharpe ratio is the only reasonable choice if the distributions in question are normal. Financial returns are not normally distributed and, as a result, the search for performance measures which take into account asymmetry and fat tails in returns distributions is of central interest to finance practitioners and academics.

The Omega function of a distribution, introduced in [2], is a function equivalent to the distribution itself. If the Omega function of distribution F dominates the Omega function of distribution G globally, then the quality of a bet at any return level for F is higher than for G . The typical case of comparing two investments is not settled in this way however because, in the generic case, the Omega functions of the distributions will cross. Even if one asset dominates another, the Omega function of a portfolio of the two will typically cross both of them.

Recently a number of practitioners and academics have discussed the Omega function and related functions as performance measures, treating the values of the Omega function at isolated points as functionals on the underlying distribution. This leads to ambiguous or incorrect conclusions in general as one needs to make use of the entire Omega function in generic situations.

Some authors have also noted the similarity in rankings by values of Omega functions at various thresholds with those produced by Sortino ratios e.g. [1],[7]. Others, like Darsinos and Satchell [5] and Farinelli and Tibiletti [6] have suggested 'generalisations' in which Sortino ratios, Omega functions and other similar constructions appear as particular instances of a one or two parameter family of functions formed from the underlying distribution.

The lottery provides a useful test case for the effectiveness of performance measures. It looks at two assets, modelled on the standard buy a ticket/sell a ticket lottery choice. For example, the choice between buying or selling the a lottery in which the payoff for the Buy is a loss of \$1 in 999,999 times in 1 million and a win of \$999,999 once in a million times. The Sell payoff is -\$999,999 once in a million times and \$1 the remaining 999,999 times.

The mean for both Buy and Sell is 0 and the variance is 999,999. The Buy and Sell lotteries are mirror images: the skewness and all odd moments are negative for the Sell and equal in magnitude but positive for the Buy. All even moments are equal for both distributions.

The lottery test simply asks if the performance measure agrees with the man in the street, who regards the Sell as unacceptably more risky than the Buy.

The purpose of this note is to point out that the Sortino ratio fails the test of ranking the lottery Buy and Sell. The 'Generalised Sharpe Ratios' described by Darsinos and Satchell also fail this test except in the case of a trivial variant of the Omega function. The remaining 'generalisations' pass the lottery test only when they apply to a strictly smaller class of distributions than those for which Omega functions are defined—a rather odd notion of generalisation. We also discuss briefly, the need for a functional on distributions as a ranking tool [4].

2 'Generalising' Omega Functions

A number of variants of the Omega function described in Keating and Shadwick [2] have recently been proposed. These consider functions which generalise the Sharpe ratio by:

a) replacing the difference between the mean return and the risk free rate by the difference between the mean return and an arbitrary return level r ,

and

b) replacing the standard deviation by a 'lower partial moment' namely the expected value of a power λ of the formal put payoff function with strike of r , $\max\{r - x, 0\}$.

$$\Psi_F(r, \lambda) := \frac{\mu_F - r}{\left(\int_A^r (r - x)^\lambda dF(x) \right)^{1/\lambda}} . \quad 2.1$$

For any positive λ , $\Psi_F(r, \lambda)$, is intended to be consistent with a degree of risk aversion determined by λ for returns below the threshold 'r' (which corresponds to Sortino's MAR or Minimum Acceptable Return) and zero absolute risk aversion above 'r'. These functions have been referred to by Darsinos and Satchell as "generalised Sharpe ratios"[5] equation 13.

While this seems reasonable for a fixed value of r, it leads to a rather schizophrenic treatment of up and downside when one considers $\Psi_F(r, \lambda)$ as a function on $[A, B]$, the support of the distribution F . For example, when $r = 1\%$ per month, one is meant to regard returns as seen through a utility function with positive absolute risk aversion below 1% and zero absolute risk aversion above 1%. To calculate the function at $r = 0.5\%$, the returns between 0.5% and 1% are now treated as upside and hence viewed with zero absolute risk aversion. From this alone it is clear that, except in the case where λ is 1, this approach may cause problems. As is shown below, these include perverse choices in the lottery test.

The functions $\Psi_F(r, \lambda)$ are defined formally for any positive value of the parameter λ , however if λ is greater than 1, the integral may fail to converge unless moments of order higher than 1 are finite. *Thus the class of distributions for which these functions are defined is strictly smaller than the class for which Omega functions are defined, whenever λ is greater than 1.*

While it is obvious that the class of functions defined by equation 2.1 shares some of the useful properties of Omega functions, these seem not to have been noted before. For example they have the same invariance as Omega functions [3, Theorem 1] when the distribution F is transformed by an orientation-preserving affine change of variable. They lack the symmetry property of Omega functions and hence also lack the invariance under orientation-reversing affine changes of variable. They are monotone decreasing functions which go to infinity at the left end of the support of the distribution and tend to minus 1 as r tends to infinity (independent of the support of the distribution). The mean is the unique point where they take the value zero. More importantly, it is also the case that there is a one-to-one relation between the function $\Psi_F(r, \lambda)$ and the distribution F .

In the case where the parameter λ is equal to 2, Ψ_F is the Sortino Ratio as a function of r (which plays the role of MAR, the minimal acceptable return, in Sortino's notation).

In the case where λ is equal to 1, Ψ_F is simply the Omega function of the distribution F with the constant 1 subtracted as the Theorem 1 of [3] shows. It is clear that Omega functions are only naturally defined up to a positive affine transformation, i.e. if Ω_F is the Omega function any function $a \Omega_F + b$ will provide the same ranking at any point r, provided that the constant a is positive. This amounts to the difference between temperature in Fahrenheit and Celsius scales. Any such modification is clearly trivial. This has not prevented them from entering the academic literature (see H. Kazemi et al, "Omega as a Performance Measure", 2003, Working Paper CISDM, which produced "a better Omega" also called "Sharpe Omega" by subtracting 1 from the original). Trivial modifications will not be considered further here.

3 The Lottery test for 'Generalised' Omega Functions

Like Omega functions, the functions defined in equation 2.1 provide an alternative picture of the distribution F . In the case of Omega functions the utility of this picture is clear: Omega functions are immediately informative about asymmetry and fat tails in distributions. The utility of the pictures provided by $\Psi_F(r, \lambda)$ is less obvious.

In fact, the lottery test shows that, except in the case $\lambda=1$, these functions distort the comparison of the Buy and Sell lotteries to the extent that they fail to produce the common sense verdict of the dominance of Buy over Sell. The same is true of the cases for λ in the interval (0, 1), so that the value $\lambda=1$ is actually critical for a correct ranking of the lotteries.

The figures below show typical behaviour for the difference $\Psi_{Buy}(r,\lambda) - \Psi_{Sell}(r,\lambda)$ with λ in the interval $(0,2)$ and greater than or equal to 2 for the lottery corresponding to a \$999,999 payoff one time in a million.

For λ in the interval $(0,1)$ the asymptotic preference is the correct one, namely Buy over Sell. Figure 1a shows that there is an open interval over which this preference is reversed below the mean. For λ in the interval $(1,2)$, the correct preference below the mean is reversed above the mean. The value $\lambda = 1$ corresponding to the Omega function is critical: no other value produces the correct answer both above and below the mean.

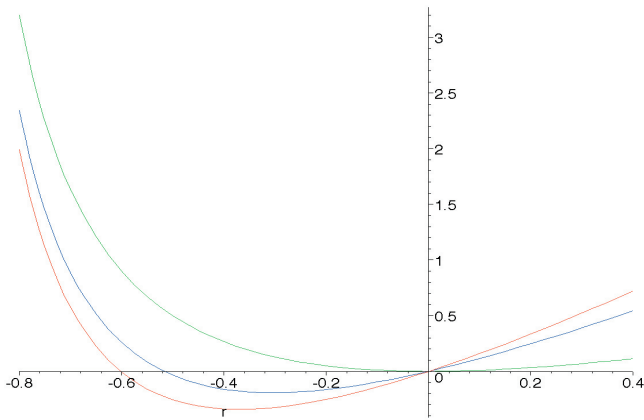


Figure 1 a. The difference $\Psi_{Buy}(r,\lambda) - \Psi_{Sell}(r,\lambda)$ for $\lambda = 75/80$ (red), $\lambda = 95/100$ (blue) and $\lambda = 1$ (green).

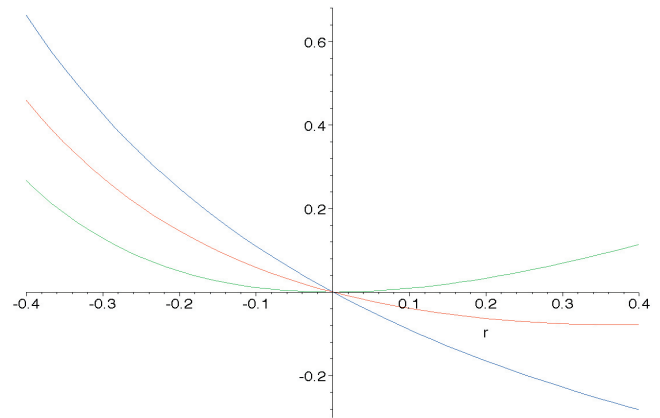


Figure 1 b. The difference $\Psi_{Buy}(r,\lambda) - \Psi_{Sell}(r,\lambda)$ for $\lambda = 105/100$ (red), $\lambda = 2$ (blue) and $\lambda = 1$ (green).

This result is independent of the particular choice of lottery. As Omega functions and the functions $\Psi_F(r,\lambda)$ all have the obvious invariance property for positive affine transformations of the variable r , it is enough to consider the lottery up to affine transformations. Note that, up to a positive re-scaling to set the ticket price at \$1, all lotteries are described simply by the probability 'a' of a loss. The exact formula for $\Psi_{Buy}(r,\lambda)$ and $\Psi_{Sell}(r,\lambda)$ as a function of this probability, r and λ is easily obtained and the reader may verify that the perverse choices shown above occur for all lotteries.

4 Ranking Distributions

The lottery is a simple test which these 'generalisations' of Omega functions all fail. This makes it clear that it is dangerous to apply them in cases where there is no alternative route to obtaining the correct ranking of two distributions.

In fact, any attempt to rank distributions purely on the basis of the values of their Omega functions will fail whenever the Omega functions cross—which is the generic case in financial applications. This problem is signalled by the point at which the Buy and Sell Omega functions produce the same value, namely the common mean. A small increase in the mean of the Sell lottery would produce two crossings and an interval over which the simple-minded ranking approach would reverse the choice of Buy over Sell. As the mean of the Sell lottery is increased the size of this interval will also increase, as will the attractiveness of the Sell, in spite of its left bias.

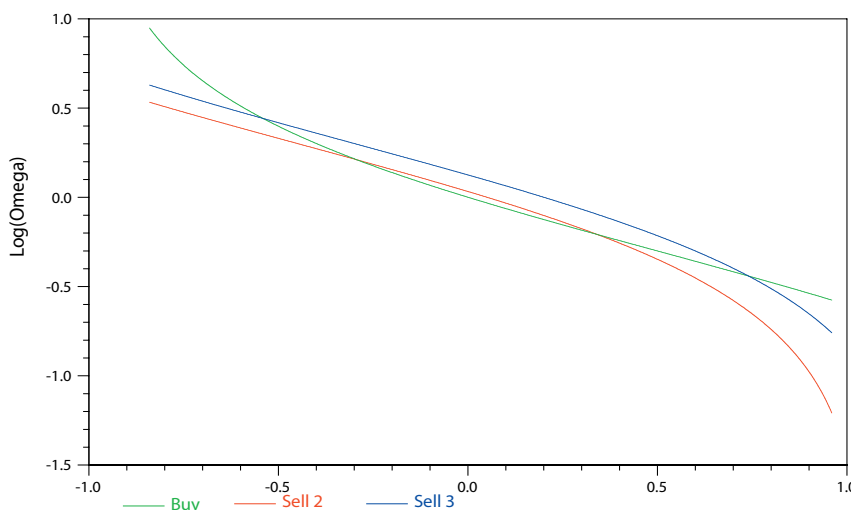


Figure 2. The green curve is $\log(\text{Omega})$ for the original Buy lottery. The red is for a Sell lottery, Sell 2 whose distribution is obtained from the Sell lottery by translating it to the right. The slightly higher mean produces an open interval over which the Omega function of Sell 2 dominates that of Buy. This interval is extended, and the dominance increased, by translating the Sell lottery further to the right to produce Sell3 (in blue).

This shows clearly that any attempt to rank these distributions by their Omega values at fixed points will fail. One must take into account the up and downside trade-offs and the mean return as well. For almost anyone, Buy would be preferable to Sell2 but many would find Sell 3 preferable to Buy.

What is needed to deal with this (generic) problem is a functional which provides a ranking of distributions on the basis of the entire distribution (or its proxy, the Omega function) Cascon and Shadwick [4]. This is a true generalisation of the Omega function performance measurement introduced in [2].

References

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